ZUBOV'S METHOD FOR INTERCONNECTED SYSTEMS - A DISSIPATIVE FORMULATION*

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Abstract. We study the domain of attraction of an asymptotically stable fixed point of the feedback interconnections of two nonlinear systems. For each subsystem we introduce an auxiliary system and assume that it is uniformly locally asymptotically stable at the origin. It is then shown that each subsystem is integral input-to state stable(iISS) regarding the state of the other subsystem as input. If the interconnection of the two subsystems satisfies a small gain condition, an estimate for the domain of attraction of the whole system may be obtained by constructing a nonsmooth Lyapunov function.

Key words. nonlinear systems, input-to state stability(ISS), integral input-to state stability (iISS), ISS Lyapunov function, small gain condition, Zubov's method

1. Introduction.

We consider the problem of approximating the domain of attraction of an asymptotically stable fixed point for interconnected systems and extend the results on Zubov's method obtained in [6, 8]. In particular, we partly remove the condition that each of the subsystems has to be input-to-state stable (ISS). The approach presented here provides the framework for the numerical construction of dissipative iISS Lyapunov functions as viscosity solutions of a Zubov type equation. With this approach such iISS Lyapunov functions may be constructed for subsystems of interconnected systems. An overall Lyapunov function for the interconnection can then be obtained using a small gain type approach. The benefit of the procedure is that it avoid the numerical cost associated with solving a Zubov equation in a higher dimensional space; at the price of only obtaining conservative estimates for the domain of attraction. The extension of the ideas to interconnections of several subsystems is the object of current investigation.

1.1. Preliminaries.

In this paper the Euclidean norm in $\mathbb{R}^n$ is denoted by $\|\cdot\|$ and $B(z,r) := \{x \in \mathbb{R}^n \mid \|x-z\| < r\}$ denotes the set of points with distance less than r from z. Let $\mathbb{R}_+$ denote the interval $[0,+\infty)$. A continuous function $\alpha : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is said to be positive definite if it satisfies $\alpha(0) = 0$ and $\alpha(s) > 0$ for all $s > 0$. A positive definite function is of class $\mathcal{K}$ if it is strictly increasing; and of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and unbounded. A continuous function $\beta : \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{L}$ if it is strictly decreasing to 0 as $r \to \infty$; and we call a continuous function β with two real nonnegative arguments of class $\mathcal{KL}$ if it is of class $\mathcal{K}_\infty$ in the first and of class $\mathcal{L}$ in the second argument.

Consider a system of the form

$$\dot{x} = f(x,u)\,. \tag{1.1}$$

satisfying standard regularity assumptions, $x \in \mathbb{R}^n, u \in \mathbb{R}^m$. If there exists a proper, positive definite function $V : \mathbb{R}^n \to \mathbb{R}_+$ such that along solutions of (1.1) we have

$$\dot{V}(x) \leq -\alpha(\|x\|) + \beta(\|u\|) \tag{1.2}$$

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then system (1.1) is called ISS if α, β may be chosen to be in $\mathcal{K}_\infty$ and integral ISS (iISS), if α is positive definite and $\beta \in \mathcal{K}$, see [11, 5, 10].

In this paper, we consider a system of two interconnected systems without control (or perturbation). In the following $n_1, n_2 \in \mathbb{N}$ denote the dimension of the two subsystems and we let $N := n_1 + n_2$. Given $f_i : \mathbb{R}^N \to \mathbb{R}^{n_i}$, $i = 1, 2$, we consider the system

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2) \\ \dot{x}_2 = f_2(x_2, x_1) \\ x_1(0) = x_1^0,\, x_2(0) = x_2^0 \end{cases} \tag{1.3}$$

We let $F = (f_1, f_2) : \mathbb{R}^N \to \mathbb{R}^N$, $x = (x_1, x_2)$, $x^0 = (x_1^0, x_2^0)$ so that we may equivalently write the system as

$$\begin{cases} \dot{x} = F(x) \\ x(0) = x^0 \end{cases} \tag{1.4}$$

It is assumed that F is locally Lipschitz continuous on $\mathbb{R}^n$, has a fixed point in $x^* = 0$ and that the fixed point is locally exponentially stable. Further $\varphi(t, x^0)$ denotes the solution of (1.3) at time t corresponding to the initial solution $x(0) = x^0$. We are interested in the *domain of attraction* of the fixed pointed $x^* = 0$, that is the set

$$\mathcal{D} = \left\{ x^0 \in \mathbb{R}^N : \varphi(t, x^0) \to 0 \quad \text{for } t \to \infty \right\} . \tag{1.5}$$

In order to get the domain of attraction of the system 1.3, firstly we need to get dissipative iISS Lyapunov functions for each subsystem of the system.

1.2. The auxiliary system. To construct dissipative iISS Lyapunov functions for one of the subsystems of (1.3), say $f := f_i$, we fix an auxiliary function η, a bound $R > 0$ and consider

$$\dot{x} = f_\eta(x, u) := f(x, u) - \eta(\|u\|)x\,, \quad \|u\| \le R\,, \tag{1.6}$$

where η is a $\mathcal{K} \cup \{0\}$ function and locally Lipschitz on $(0, \infty)$. The admissible input values are in $U_R := \operatorname{cl} B(0, R)$ and the control inputs are $u \in \mathcal{U}_R := \{u : \mathbb{R} \to \mathbb{R}^m \mid \|u\|_\infty \le R\}$, where $\|\cdot\|_\infty$ denotes the infinity, i.e. the essential supremum of a control function u.

To explain the significance of the auxiliary system assume for the moment that we have a *robust Lyapunov function* v for (1.6), that is, a Lyapunov function such that for all $\|u\| \le R$ we have for some positive definite function α

$$\nabla v(x) f_\eta(x, u) \le -\alpha(\|x\|)\,. \tag{1.7}$$

Then by the definition of f_η we obtain $\nabla v(x) f(x, u) \le -\alpha(\|x\|) + \nabla v(x)\eta(\|u\|)\|x\|$. If we find a positive definite α and a $\mathcal{K}$ function β satisfying

$$\nabla v(x) f(x, u) \le -\alpha(v_\eta(x)) + \beta(\|u\|)\,, \tag{1.8}$$

then each subsystem of the system (1.3) is iISS.

2. Main results.

2.1. The domain of attraction of the auxiliary system. In this section we study the properties of the system (1.6). We now consider the perturbed system

$$\dot{x} = f_\eta(x,u) := f(x,u) - \eta(\|u\|)x\,, \quad \|u\| \le R\,, \tag{2.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $\eta \in \mathcal{K} \cup \{0\}$ is locally Lipschitz on $(0,\infty)$. Under these assumptions local Lipschitz continuity of $f_\eta(x,u)$ is guaranteed on $\mathbb{R}^n \backslash \{0\}$, which is sufficient as we assume uniform asymptotic stability in $x^* = 0$. The solutions of (2.1) are denoted by $x_\eta(\cdot, x^0, u)$.

We assume that $x^* = 0$ is uniformly locally asymptotically stable for (2.1), i.e.

(H1) there exists a constant $r > 0$ and a function β of class $\mathcal{KL}$ such that $\|x_\eta(t,x^0,u)\| \le \beta(\|x^0\|,t)$ for any $x^0 \in B(0,r)$, any $u \in \mathcal{U}_R$, and all $t \ge 0$.

By Sontag's $\mathcal{KL}$ lemma (see [5]) for any $\beta \in \mathcal{KL}$ there exist two functions $\gamma_1, \gamma_2 \in \mathcal{K}_\infty$ such that

$$\beta(r,t) \le \gamma_2(\gamma_1(r)e^{-t}). \tag{2.2}$$

In the sequel we will work primarily with the functions $\gamma_1, \gamma_2 \in \mathcal{K}_\infty$.

Under the assumption of uniform asymptotic stability of the origin $x^* = 0$ for (2.1), we define the corresponding robust domain of attraction as follows.

Definition 2.1. *The* (uniform) robust domain of attraction *of* (2.1) *is*

$$\mathcal{D}_\eta = \left\{ x^0 \in \mathbb{R}^n : \begin{array}{l} \text{there exists } \mu \in \mathcal{L} \text{ such that } \|x_\eta(t,x^0,u)\| \le \mu(t) \\ \text{for all } t > 0,\ u \in \mathcal{U}_R \end{array} \right\}.$$

The following properties of $\mathcal{D}_\eta$ are shown in [6, Proposition 2.3].

Proposition 2.2. *Consider system* (2.1) *and assume (H1), then*

(i) $\mathcal{D}_\eta$ *is an open, connected, invariant set with* $\operatorname{cl} B(0,r) \subset \mathcal{D}_\eta$.

(ii) $\sup_{u\in\mathcal{U}_R} \{t(x,u)\} \to +\infty$ *for* $x \to x^0 \in \partial\mathcal{D}_\eta$ *or* $\|x\| \to \infty$, *where* $t(x,u) = \inf\{t > 0 : x_\eta(t,x,u) \in B(0,r)\}$.

(iii) $\operatorname{cl}\mathcal{D}_\eta$ *is an invariant set which is contractible to* 0.

(iv) If for some $\|u_0\| \le R$, $f_\eta(\cdot,u_0)$ *is of class* C^1, *then* $\mathcal{D}_\eta$ *is* C^1*-diffeomorphic to* $\mathbb{R}^n$.

It is shown in [9, 6, 7] how a Zubov type equation may be formulated that allows for the computation of a maximal Lyapunov function on the domain of attraction. In this paper we use the formulation of [7], which is a slight generalization of [6]. To this end an optimal control problem is defined using a running cost g, which is chosen in such a manner that the function $g : \mathbb{R}^n \times U_R \to \mathbb{R}$ is continuous and satisfies

(H2)

(i) For all $u \in U = \{u : \|u\| \le R\}$, $g(x,u) \le C\gamma_2^{-1}(\|x\|)$ for all $x \in \mathbb{R}^n$, γ_2 from (2.2) and some $C > 0$, and $g(x,u) > 0$ for $x \ne 0$.

(ii) There exists a constant $g_0 > 0$ such that $\inf\{g(x,u) \mid x \notin B(0,r),\ u \in U\} \ge g_0$.

(iii) For each $J > 0$ there exists $L_J > 0$ such that $\|g(x,u) - g(y,u)\| \le L_J\|x - y\|$ for all $\|x\|, \|y\| \le J$, and all $\|u\| \le R$.

We now introduce the value function of a suitable optimal control problem related to (2.1). Consider the functional $J : \mathbb{R}^n \times U_R \to \mathbb{R}_+ \cup \{+\infty\}$

$$J_\eta(x,u) := \int_0^{+\infty} g(x_\eta(t), u(t))dt\,,$$

and the optimal value function

$$v_\eta(x) := \sup_{u\in\mathcal{U}_R} 1 - e^{-J_\eta(x,u)}\,. \tag{2.3}$$

Since g is nonnegative it is immediate that $v_\eta(x) \in [0,1]$ for all $x \in \mathbb{R}^n$. Furthermore, standard techniques from optimal control imply that v_γ satisfies a dynamic programming principle, i.e. for each $t > 0$ we have

$$v_\eta(x) = \sup_{u\in\mathcal{U}_R} \{(1 - G(x,t,u)) + G(x,t,u)v_\eta(x(t,x,u))\} \tag{2.4}$$

with

$$G(t,x,u) := \exp\left(-\int_0^t g(x(\tau,x,u),u(\tau))d\tau\right). \tag{2.5}$$

An application of the chain rule shows $(1-G(x,t,u)) = \int_0^t G(x,\tau,u)g(x(\tau,x,u)u(\tau))d\tau$ implying

$$v_\eta(x) = \sup_{u\in\mathcal{U}_R} \left\{ \int_0^t G(x,\tau,u)g(x(\tau,x,u),u(\tau))d\tau + \ G(x,t,u)v_\eta(x(t,x,u))\right\}.$$

The next proposition shows the relation between $\mathcal{D}_\eta$ and v_η, and the continuity of v_η, [6, Proposition 3.1].

Proposition 2.3. *Assume (H1), (H2). Then*

(i) $v_\eta(x) < 1$ *if and only if* $x \in \mathcal{D}_\eta$. *(ii)* $v_\eta(x) = 0$ *if and only if* $x = 0$.

(iii) v_η *is continuous on* $\mathbb{R}^n$. *(iv)* $v_\eta(x) \to 1$ *for* $x \to x^0 \in \partial\mathcal{D}_\eta$ *and* $\|x\| \to \infty$.

We now turn to the formulation of a suitable partial differential equation for which v_η is a solution. Since in general this function is not differentiable we have to work with a more general solution concept, namely viscosity solutions. We refer to [3] for the basic concepts in this area.

Recalling that v_η is locally bounded on $\mathbb{R}^n$, by the dynamic programming principle (2.4), it follows that v_η can be characterized as the unique viscosity solution of the Zubov equation

$$\sup_{\|u\|\le R} \{Dv_\eta(x)f_\eta(x,u) + (1 - v_\eta(x))g(x,u)\} = 0\,. \tag{2.6}$$

The following Theorem 2.4, Proposition 2.5 are the same as Theorem 3.8 and Theorem 4.1 in [6]. Theorem 2.6 is proved in a similar way to the proof of Proposition 4.2, Proposition 4.3 in [6].

Theorem 2.4. *Consider the system* (2.1) *and a function* $g : \mathbb{R}^n \times U_R \longrightarrow \mathbb{R}$ *such that (H1) and (H2) are satisfied. Then* (2.6) *has a unique bounded and continuous viscosity solution* v_η *on* $\mathbb{R}^n$ *satisfying* $v_\eta(x) = 0$ *for* $x = 0$.
This function coincides with v_η *from* (2.3). *In particular the characterization* $\mathcal{D}_\eta = \{x \in \mathbb{R}^n \mid v_\eta(x) < 1\}$ *holds.*

Proposition 2.5. *Assume (H1) and (H2) and consider the unique viscosity solution* v_γ *of* (2.6) *with* $v_\eta(0) = 0$. *Then the function* v_η *is a robust Lyapunov function for the system* (2.1) *on* $\mathcal{D}_\eta$. *More precisely we have*

$$v_\eta(x(t,x^0,u)) - v_\eta(x^0) \le \left[1 - e^{-\int_0^t g(x(\tau),u(\tau))d\tau}\right](v_\eta(x(t,x^0,u)) - 1) < \ 0$$

for all $x^0 \in \mathcal{D}_\eta \setminus \{0\}$ *and all functions* $u \in \mathcal{U}_R$.

Now we turn to the Lipschitz property of v_η.

THEOREM 2.6. *Assume (H1) and (H2) and consider the unique viscosity solution* v_η *of* (2.6) *with* $v_\eta(x) = 0$ *for all* $x \in \mathcal{D}_\eta$. *Assume that* $f_\eta(\cdot, u)$ *and* $g(\cdot, u)$ *are uniformly Lipschitz continuous in* $\mathcal{D}_\eta$, *with constants* L_f, $L_g > 0$ *uniformly in* $u \in U_R$, *and assume that there exists an open neighborhood* W *of* 0 *such that for all* x, $y \in W$ *the inequality*

$$|g(x,u) - g(y,u)| \leq K\gamma_2^{-1}(\max\{\|x\|, \|y\|\})^s \|x - y\|$$

holds for some $K > 0$, $s > L_f$ *and* γ_2 *from* (2.2). *Then* v_η *is locally Lipschitz continuous in* $\mathbb{R}^n$.

Proof. The basic idea of the proof is the same as the proof of Proposition 4.2 and Proposition 4.3 in [6]. □

Under the conditions of (H1) and (H2), $v_\eta(x)$ is the unique viscosity solution of (2.6). Therefore $p \in D^+ v_\eta(x)$, the superdifferential of v_η at x, satisfies

$$\sup_{\|u\| \leq R} \{p \cdot f_\eta(x,u) + (1 - v_\eta(x))g(x,u)\} \leq 0, \tag{2.7}$$

where $f_\eta(x,u)$ is defined in (2.1).
Using the definition of f_η, we conclude v_η is a viscosity subsolution of

$$Dv_\eta(x)f(x,u) \leq -(1 - v_\eta(x))g(x,u) + Dv_\eta(x)x\ \eta(\|u\|)\,, \tag{2.8}$$

on $\mathcal{D}_\eta$ with the constraint $\|u\| \leq R$.

As we have seen that with a reasonable choice of g we can force v_η to be locally Lipschitz, we will use a bound on the Lipschitz constant. For simplicity, let

$$L(x) := \inf\{L \geq 0 \mid L \text{ is a Lipschitz constant for } v_\eta \text{ on } B(x,1)\}\,. \tag{2.9}$$

PROPOSITION 2.7. *Let the conditions of Theorem 2.6 hold and assume that* $L(x)\|x\|$ *is bounded on* $\mathbb{R}^n$. *Then there exist a positive definite function* α *and* $\beta \in \mathcal{K}$ *such that*

$$Dv_\eta(x)f(x,u) \leq -\alpha(v_\eta(x)) + \beta(\|u\|) \tag{2.10}$$

holds almost everywhere on $\mathcal{D}_\eta$ *under* $\|u\| \leq R$. *Hence* v_η *is a local viscosity iISS Lyapunov function for the system*

$$\dot{x} = f(x,u)\,, \quad x \in \mathcal{D}_\eta, \|u\| \leq R\,. \tag{2.11}$$

Proof. From Theorem 2.6, v_η is a Lipschitz continuous function in $\mathbb{R}^n$ with Lipschitz constant L_v. Thus $v_\eta(x)$ is differentiable almost everywhere, and $D^- v_\eta(x)$, $D^+ v_\eta(x)$ are contained in $\overline{B}(0, L_v)$ for all $x \in \mathbb{R}^n$. Because v_η is a viscosity subsolution of (2.8), we get

$$Dv_\eta(x)f(x,u) \leq -(1 - v_\eta(x))g(x,u) + Dv_\eta(x)x\ \eta(\|u\|)$$

holds almost everywhere on $\mathcal{D}_\eta$.
As $(1 - v_\eta(x))g(x,u)$ is nonnegative on Dv_η, we may choose α positive definite such that

$$\alpha(v_\eta(x)) \leq (1 - v_\eta(x))g(x,u), \quad \text{on } Dv_\eta \times U_R. \tag{2.12}$$

Furthermore, by assumption there exists a uniform bound for $Dv_\eta(x)x$, wherever v_η is differentiable. As also $\eta(\|u\|) \in \mathcal{K} \cup \{0\}$, there exists $\beta \in \mathcal{K}$ satisfying

$$\beta(\|u\|) \geq Dv_\eta(x)x\ \eta(\|u\|), \quad \text{for } \|u\| \leq R. \tag{2.13}$$

Hence, as desired,

$$Dv_\eta(x)f(x,u) \leq -\alpha(v_\eta(x)) + \beta(\|u\|)\,, \tag{2.14}$$

holds for all points of differentiability of v_η lying in D_η. □

In the next section, we use to results on the auxiliary system to study the coupled system (1.3).

2.2. Coupled Systems. Under the assumption that the corresponding auxiliary system for each subsystem of (1.3) is uniformly locally asymptotically stable at the origin, each subsystem is iISS and v_i is the iISS Lyapunov function for the subsystem i on $x_i \in \mathcal{D}_{\eta_i}$ $(i = 1, 2)$.

If Proposition 2.7 is applicable, we choose α_i positive definite and $\beta_i \in \mathcal{K}$ such that

$$\dot{v}_i(x_i) \leq -\alpha_i(v_i(x_i)) + \beta_i(\|x_j\|), \tag{2.15}$$

hold for almost all $x_i \in \mathcal{D}_{\eta_i}$ and $\|x_j\| \leq R_j$, $i, j = 1, 2$ and $i \neq j$.
By comparing $\|x_j\|$ and $v_j(x_j)$, we may construct σ_i, $i = 1, 2$ such that

$$\dot{v}_i(x_i) \leq -\alpha_i(v_i) + \sigma_i(v_j), \tag{2.16}$$

hold for $x_i \in \mathcal{D}_{\eta_i}$ and $\|x_j\| \leq R_j$, $i, j = 1, 2$ and $i \neq j$.

Based on this formulation of the ISS property a Lyapunov function can be obtained for the coupled system. To this end we need the following small gain result. We define the matrix

$$\Gamma = \begin{pmatrix} 0 & \sigma_1 \\ \sigma_2 & 0 \end{pmatrix},$$

which defines a monotone map $\Gamma : \mathbb{R}^2_+ \longrightarrow \mathbb{R}^2_+$ by

$$\Gamma(s) := (\sigma_1(s_2), \sigma_2(s_1))^{\mathbb{T}}\,,\ s \in \mathbb{R}^2_+\,. \tag{2.17}$$

Furthermore define the diagonal operator $A : \mathbb{R}^2_+ \longrightarrow \mathbb{R}^2_+$

$$A(s) := (\alpha_1(s_1), \alpha_2(s_2))^{\mathbb{T}}\,,\ s \in \mathbb{R}^2_+\,, \tag{2.18}$$

and a diagonal operator $E : \mathbb{R}^2_+ \longrightarrow \mathbb{R}^2_+$, which is defined through two $\mathcal{K}_\infty$ functions φ_1, φ_2 by $E(s) := ((\mathrm{id} + \varphi_1)(s_1), (\mathrm{id} + \varphi_2)(s_2))$. With this information we can formulate the following small-gain theorem on $\mathcal{D}_{\eta_1} \times \mathcal{D}_{\eta_2}$.

Theorem 2.8. *Consider the coupled system* (1.3) *and assume that for each of the subsystems there exists an ISS Lyapunov function v_i, $i = 1, 2$ in the sense of* (2.16). *If there exists a diagonal operator E such that the small gain condition*

$$E \circ \Gamma \circ A^{-1}(s) \not\geq s\,, \forall s \in \mathbb{R}^2_+ \setminus \{0\}\,, \tag{2.19}$$

is satisfied, then there exists a continuously differentiable path $\theta : [0, \infty) \to \mathbb{R}^2$, such that $\theta(0) = 0$ and θ' is bounded and positive so that

$$E \circ \Gamma \circ A^{-1}(\theta(s)) < \theta(s)\,, \quad \forall s \in (0, \infty)\,. \tag{2.20}$$

Assume further that there exist two constants c, C such that

$$0 < c < \frac{d}{ds}\theta_i^{-1} \circ \alpha_i(s) < C, \ \forall s \in (0, \infty)\,. \tag{2.21}$$

A Lyapunov function for the coupled system is then given by

$$v(x_1, x_2) := \max\left\{\theta_1^{-1} \circ \alpha_1(v_1(x_1)), \theta_2^{-1} \circ \alpha_2(v_2(x_2))\right\}. \tag{2.22}$$

Proof. This theorem is the small-gain Theorem 4.5 in [4]. Therefore we omit the details of the proof here. □

Since we only have local version of the Lyapunov functions we need a local version suitable for our case. The procedure we now propose is the following:

(i) For each of the subsystems $i = 1, 2$ choose $\eta_i \in \mathcal{K} \cup \{0\}$ and compute the maximal Lyapunov function v_i on $\mathcal{D}_{\eta_i}$ by solving the corresponding Zubov equation (2.6).
(ii) For each v_i compute the corresponding α_i, σ_i
(iii) Check that $\Gamma \circ A^{-1}(s)$ satisfies the small gain condition (2.19).
(iv) If this is the case choose the path θ along which $E \circ \Gamma \circ A^{-1}$ is decreasing as in (2.20).[1]
(v) Check that there exist two constants satisfy $0 < c < \frac{d}{ds}\theta_i^{-1} \circ \alpha_i(s) < C$, $\forall s \in (0, 1)$.
(vi) Define

$$v(x_1, x_2) := \max\left\{\theta_1^{-1} \circ \alpha_1(v_1(x_1)), \theta_2^{-1} \circ \alpha_2(v_2(x_2))\right\}, \tag{2.23}$$

and let $\tau := \min\left\{\theta_1^{-1} \circ \alpha_1(1), \theta_2^{-1} \circ \alpha_2(1)\right\}$.

We claim that provided all steps in the construction can be completed successfully then with this choice of v we have that $v^{-1}([0, \tau))$ is a subset of the domain of attraction of the coupled system. This is the gist of the following theorem.

THEOREM 2.9. *Consider the coupled system* (1.3)*. Assume for each of the subsystems a solution v_i of the Zubov equation* (2.6) *is available. Assume furthermore, that $\Gamma \circ A^{-1}(s)$ satisfies the small gain condition* (2.19) *and there exist two constants satisfying*

$$0 < c < \frac{d}{ds}\theta_i^{-1} \circ \alpha_i(s) < C\,, \quad \forall s \in (0, 1)\,.$$

Then the function v defined in (2.22) *is a local Lyapunov function for* (1.3) *in $x^* = 0$. With $\tau := \min\left\{\theta_1^{-1} \circ \alpha_1(1), \theta_2^{-1} \circ \alpha_2(1)\right\}$ we have $\mathcal{D}_v := v^{-1}([0, \tau)) \subset \mathcal{D}$ from* (1.5).

Proof. Without loss of generality we may assume that $\tau = 1$, because we can always rescale the path θ. We first note, that the choice of τ ensures, that $v(x_1, x_2) < \tau = 1$ implies that $x_1 \in D_{\eta_1}$ and $x_2 \in D_{\eta_2}$, because of Proposition 2.3 (i).

Let $x \in \mathcal{D}_v \setminus \{0\}$. We assume first that for a given $x = (x_1, x_2)$ we have $v(x_1, x_2) = \theta_1^{-1} \circ \alpha_1(v_1(x_1)) > \theta_2^{-1} \circ \alpha_2(v_2(x_2))$.

We now denote $z_i = \alpha_i(v_i(x_i))$, $i = 1, 2$.

$$\begin{aligned} \dot{v}_1(x_1) &\leq -\alpha_1(v_1(x_1)) + \sigma_1(v_2(x_2)) \\ &= -\theta_1 \circ \theta_1^{-1}(z_1) + \sigma_1 \circ \alpha_2^{-1}(z_2) \\ &\leq -\theta_1 \circ \theta_1^{-1}(z_1) + \sigma_1 \circ \alpha_2^{-1} \circ \theta_2 \circ \theta_1^{-1}(z_1) \end{aligned} \tag{2.24}$$

[1] This can be done numerically in a simple manner, see [2].

Now for $\xi = \theta_1^{-1}(z_1)$ we have by (2.20)

$$
\begin{aligned}
E \circ \Gamma \circ A^{-1}(\theta(\xi)) &< \theta(\xi), \\
\Gamma \circ A^{-1}(\theta(\xi)) &< E^{-1} \circ \theta(\xi)
\end{aligned} \tag{2.25}
$$

Then we obtain

$$
\begin{aligned}
\dot{v}_1(x_1) &\leq -\theta_1 \circ \theta_1^{-1}(z_1) + \sigma_1 \circ \alpha_2^{-1} \circ \theta_2 \circ \theta_1^{-1}(z_1) \\
&< ((\mathrm{id} + \varphi_1)^{-1} - \mathrm{id}) \circ \alpha_1(v_1(x_1)) < 0.
\end{aligned} \tag{2.26}
$$

Hence

$$
\begin{aligned}
\dot{v} &= (\theta_1^{-1} \circ \alpha_1)'(v_1(x_1))\dot{v}_1(x_1) \\
&\leq c((\mathrm{id} + \varphi_1)^{-1} - \mathrm{id}) \circ \alpha_1(v_1(x_1)) < 0.
\end{aligned} \tag{2.27}
$$

The same argument applies vice versa if $v(x_1, x_2) = \theta_2^{-1} \circ \alpha_2(v_2(x_2)) > \theta_1^{-1} \circ \alpha_1(v_1(x_1))$. This shows that the decay condition holds almost everywhere on $\mathcal{D}_v$. This implies the assertion. □

Conclusions. In this paper we have outlined a procedure for the approximation of the domain of attraction of coupled systems. In future work we aim to refine the approach. In particular, we believe that conditions can be given for g so that the assumptions of Proposition 2.7 are automatically satisfied. Also it is our aim to extend the approach to the coupling of more than two subsystems.

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