Consensus in Switching Networks with Sectorial Nonlinear Couplings. *

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1 Introduction.

Nowadays the algorithms for controlled synchronization or consensus [8, 10] in multi-agent networks are attracting considerable attention of the research community because the phenomena of synchronism and agreement, achieved via local interactions between the agents, underlie many natural phenomena and engineering solutions.

Most widespread consensus algorithms are linear ones studied for continuous and sampled time agents in [8, 10, 11] etc. A number of applications however require nonlinear couplings between the agents, e.g. problems of synchronization in oscillator networks with periodic couplings or coordination with range-restricted communication where nonlinear control laws are used to preserve the group connectivity [6, 7]. Dealing with linear couplings, one has to take into account small nonlinear distortions in measurements and quantizing effects to provide robustness. Those challenges motivated the rapid development of nonlinear consensus theory.

Most of existing results on consensus with nonlinear couplings concern special types of agents (passive agents, multiple integrators, etc.) and fixed or switching with positive dwell-time topology [2, 7, 10]. In this paper, those restrictions are overcome, with addressing agents of arbitrary order and assuming the switching topology to be only measurable function of time. We consider two important types of protocols. The first kind of protocols require undirected interaction topology and have nonlinear couplings satisfying symmetry conditions resembling the Newton’s Third Law. The second class consists of algorithms with linear couplings, satisfying some balance condition; in this case, the interaction graph may be directed. In

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both cases, the couplings may be time-variant and uncertain, satisfying a sector condition with known slopes [5]. Using the absolute stability approach, a condition for the robust consensus (in corresponding class of uncertain couplings) is obtained. The proposed criterion generalizes a number of known results on consensus and synchronization in nonlinearly coupled networks, is among the first addressing the issue of robustness and gives new results even for low-order agents. It extends the results on convergence of nonlinear consensus algorithms from [2] to the case of non-passive linear agents and uncertain time-variant couplings and the results by W. Ren [9] on synchronization of harmonic oscillators. New conditions of consensus among double-integrator agents are obtained. Unlike [1,10], this paper gives conditions for not a special type of nonlinear coupling but for the entire class of nonlinear uncertain couplings satisfying the sector condition. New results on high-gain consensus among strictly passifiable (in the sense of [4]) agents are also offered.

2 Problem statement.

2.1 Graph-theoretic preliminaries

A (directed) graph is a pair \( G = (V, E) \) of finite sets \( V \) and \( E \subset V \times V \), called the sets of nodes and arcs, respectively. The graph is undirected if \((v, w) \in E \iff (w, v) \in E\). A sequence of nodes \( v_1, \ldots, v_k \) with \((v_i, v_{i+1}) \in E \forall i\) is called the path between \( v_1 \) and \( v_k \). Node \( v \) is connected to \( w \) if \((v, w) \in E\); the graph is strongly connected if it contains a path between any two different nodes.

The symbol \( G_N \) stands for the class of all graphs \( G = (V_N, E) \) with the node set \( V_N = \{1, 2, \ldots, N\} \) that contain no self-loops: \((v, v) \notin E \forall v \in V_N\). For \( G \in G_N\), the adjacency \( (a_{jk}(G)) \) and Laplacian \( L(G) \) matrices are given by \( a_{jk}(G) := 1 \) if \((k, j) \in E\) and 0 otherwise, and

\[
L(G) := \begin{bmatrix}
\sum_{j=1}^{N} a_{1j} & -a_{12} & \cdots & -a_{1N} \\
-a_{11} & \sum_{j=1}^{N} a_{2j} & \cdots & -a_{2N} \\
\vdots & \vdots & \ddots & \vdots \\
-a_{N1} & -a_{N2} & \cdots & \sum_{j=1}^{N} a_{Nj}
\end{bmatrix}.
\]

(1)

We also put \( \Upsilon := \{z \in \mathbb{R}^N : z_k \neq z_j \text{ for some } j, k\} \) and

\[
\lambda_2(G) = \min_{z \in \Upsilon} \frac{\sum_{i,j=1}^{N} a_{ij}(G)(z_j - z_i)^2}{\sum_{i,j=1}^{N} (z_j - z_i)^2},
\]

where min is necessarily attained and the quantity (2) is called the algebraic connectivity of \( G \) [3]. The notation is motivated by the fact that for undirected graphs, (2) is the second term in the ascending sequence of eigenvalues \( \lambda_1(G) = 0 \leq \lambda_2(G) \leq \ldots \leq \lambda_N(G) \) of the symmetric matrix \( L(G) \geq 0 \), and \( \lambda_2(G) > 0 \) if and only if \( G \) is strongly connected [8]. Moreover, for such graphs [3]

\[
\lambda_2(G) \geq 2e(G) \left(1 - \cos \frac{\pi}{N}\right),
\]

(3)
where \( e(G) \geq 1 \) is the minimal number of arcs one has to delete to break the graph connectivity. A similar estimate holds for arbitrary strongly connected directed graphs \( G \in \mathcal{G}_N \). Indeed, let us introduce the undirected graph \( \hat{G} \in \mathcal{G}_N \) whose arcs are those \((v, w)\) of \( G \) plus all inverted arcs \((w, v)\). Since \( a_{jk}(G) + a_{kj}(G) \geq a_{jk}(\hat{G}) \), we have \( \lambda_2(G) \geq \frac{1}{2} \lambda_2(\hat{G}) \) by (2) and so

\[
\lambda_2(G) \geq e(\hat{G}) \left( 1 - \cos \frac{\pi}{N} \right) \geq 1 - \cos \frac{\pi}{N}. \tag{4}
\]

### 2.2 System description.

We consider a group of identical agents governed by a controllable and observable SISO state-space model

\[
\dot{x}_j(t) = Ax_j(t) + Bu_j(t), \quad y_j(t) = Cx_j(t), \quad t \geq 0. \tag{5}
\]

Here \( x_j(t) \in \mathbb{R}^d \), \( u_j(t), y_j(t) \in \mathbb{R} \) stand respectively for the state vector, control and output signal of \( j \)-th agent. Let the network topology (i.e. the pattern of communication links) at time instant \( t \geq 0 \) be defined by a graph \( G(t) = (V_N, E(t)) \in \mathcal{G}_N \). Namely, \( k \)-th agent submits its output to \( j \)-th one if and only if \((k, j) \in E(t)\). We assume \( G(.) \) to be a Lebesgue measurable function \( (G^{-1}(\Gamma)) \subset \mathbb{R} \) is a Lebesgue measurable set for any graph \( \Gamma \in \mathcal{G}_N \).

We examine control algorithms of the following commonly used type:

\[
u_j(t) = \sum_{k=1}^{N} a_{jk}(G(t)) \varphi_{jk}(t, y_k(t) - y_j(t)) \forall j. \tag{6}\]

The functions \( \varphi_{jk} : [0; +\infty) \times \mathbb{R} \to \mathbb{R} \) (possibly uncertain) are called couplings.

**Definition 1.** The protocol (6) provides output (respectively, state) consensus for the group of agents (5) if \( y_j(t) - y_k(t) \to 0 \) or, respectively, \( x_j(t) - x_k(t) \to 0 \) as \( t \to +\infty \) for any \( j, k \) and initial states \((x_j(0))_{j=1}^{N}\).

**Remark 2.** Output and state consensus are equivalent if

\[
\lim_{y \to 0} \sup_{t \geq 0} |\varphi_{jk}(t, y)| = 0 \quad \forall j, k. \tag{7}\]

Indeed, due to (6), the output consensus implies that \( |u_j(t)| \leq \sum_{k \neq j} |\varphi_{jk}(t, y_k(t) - y_j(t))| \to 0 \) as \( t \to +\infty \). The functions \( X_{jk} = x_k - x_j \), \( U_{jk} = u_k - u_j \) and \( Y_{jk} = y_k - y_j \) obey the controllable and observable model \( \dot{X}_{jk} = AX_{jk} + BU_{jk} \), \( \dot{Y}_{jk} = CX_{jk} \) and \( U_{jk}(t) \to 0, Y_{jk}(t) \to 0 \) as \( t \to +\infty \), so \( X_{jk}(t) \to 0 \) q.e.d.

We now turn to the main assumption on the protocol (6).

**Assumption 3.** The closed-loop system (5), (6) has a solution defined for \( t \geq 0 \) for any initial states \((x_j(0))\). For any \( t \geq 0 \) the graph \( G(t) \) is strongly connected and at least one of the two conditions below holds:
a) The graph $G(t)$ is undirected and $\varphi_{jk}(t, y) = -\varphi_{kj}(t, -y)$ $\forall k \neq j$;

b) The couplings are linear: $\varphi_{jk}(t, y) = w_{jk}(t)y$ and $\sum_{k=1}^{N} a_{jk}(G(t)) w_{jk}(t) = \sum_{k=1}^{N} a_{kj}(G(t)) w_{kj}(t)$ for any $j = 1, 2, \ldots, N$ (gains satisfy the balance condition).

Assumption 3 implies that $\sum_{j=1}^{N} u_j = 0$ $\sum_{j=1}^{N} y_j(t) = Ce^{tA}x_0$ with $\ddot{x}_0 = \frac{1}{N} \sum_{j=1}^{N} x_j(0)$, and the output consensus means that $y_j(t) - Ce^{tA}x_0 \to 0$ as $t \to \infty$.

Besides Assumption 3 we suppose the couplings to satisfy a sector condition. Let $S[\alpha; \beta] (-\infty < \alpha < \beta \leq \infty)$ be a set of functions $\Phi : [0; +\infty) \times \mathbb{R} \to \mathbb{R}$ such that $\Phi(t, 0) = 0$ and for any compact $K \subset \mathbb{R} \setminus \{0\}$ one has

$$\alpha < a \inf_{\sigma \in K, t \geq 0} \frac{\Phi(t, \sigma)}{\sigma} \leq \sup_{\sigma \in K, t \geq 0} \frac{\Phi(t, \sigma)}{\sigma} < \beta. \quad (8)$$

In particular, the graph of the function $\Phi(t, \cdot)$ (except the origin) lies strictly between the lines $\xi = \alpha \sigma$ and $\xi = \beta \sigma$. Our aim is to give effective consensus conditions for protocols (6) with the couplings satisfying $\varphi_{jk} \in S[\alpha; \beta]$ and Assumption 3. Since the protocol (6) with $\varphi_{jk} = 0$ can not provide consensus unless $A$ is a Hurwitz matrix, no consensus criteria can be obtained in general for the case of $\alpha < 0$, $\beta > 0$. So we deal with the case of $0 \leq \alpha < \beta$ to which the case of $\alpha < \beta \leq 0$ is reduced by replacing $\alpha, \beta, B$ with $-\beta, -\alpha, -B$ respectively.

### 3 Main results.

#### 3.1 Consensus criterion.

The following theorem is the main result of the paper.

**Theorem 4.** Suppose the protocol (6) satisfies Assumption 3 and there exist $\alpha \geq 0$, $\beta \in (\alpha; +\infty]$ such that $\varphi_{jk} \in S[\alpha; \beta]$ for any $j, k$. Let $W_y(\lambda) = C(\lambda I - A)^{-1}B$,

$$\theta = \min_{t \geq 0} \lambda_2(G(t)), \gamma = \frac{1}{\beta + \alpha}, \delta = \left(\frac{1}{\alpha} + \frac{1}{\beta}\right)^{-1}. \quad (9)$$

Let (a) $\mu \in (\alpha; \beta)$ exist such that $A - \mu NBC$ is Hurwitz and (b) for $\omega \in \mathbb{R}$, $det(i\omega I - A) \neq 0$ the inequality holds

$$Re W_y(i\omega) + \delta|W_y(i\omega)|^2 + \frac{\gamma}{2(N - 1)} \geq 0. \quad (10)$$

Then $\sum_{j,k=1}^{N} |x_j(t) - x_k(t)|^2$ is bounded (uniformly for $t \geq 0$) and the output consensus is achieved. If (7) holds, the state consensus is also provided.

**Remark 5.** Condition (a) in Theorem 4 is necessary for the robust output consensus. Moreover, under such consensus, (a) holds with any $\mu \in (\alpha, \beta)$.

Indeed, the output consensus for $G(t) \equiv G_0$ with complete graph $G_0$ and $\varphi_{jk}(t, y) = \mu y$ implies the state consensus by Remark 2. Since $N$ is an eigenvalue the Laplacian $L(G_0)$, the matrix $A - \mu NBC$ is Hurwitz [8].
Remark 6. In the conditions given by Theorem 4, the network topology is concerned only by the multiplier $\theta$ in (10). The replacement of $\theta$ by its lower estimate retains sufficiency for consensus. So if the exact computation of $\theta$ is troublesome, many constructive estimates of the algebraic connectivity, like (3) or (4), may be used.

3.2 Proofs.

A key idea in the proof of Theorem 4 is a quadratic constraint (Lemma 7) for the nonlinear controller (6). Due to the Kalman-Yakubovich-Popov (KYP) lemma, this quadratic constraint implies the existence of a quadratic Lyapunov function (Lemma 8). Throughout the section we use the following notation: given a sequence of arbitrary column vectors or scalars $v_1, v_2, \ldots, v_N$, let $\bar{v} = (v_1^T, \ldots, v_N^T)^T$.

Lemma 7. Let Assumption 3 be valid, $\varphi_{jk} \in S[\alpha; \beta]$ and $\theta, \delta, \gamma$ be defined by (9). Consider Hermitian forms $F(y, u) = -Re(y^* u) - \theta|y|^2 - \delta|y|^2 - \frac{1}{2(\lambda-1)}|u|^2$ ($y, u \in \mathbb{C}$) and $S(\bar{y}, \bar{u}) = \sum_{j,k=1}^{N} F(y_j - y_k, u_j - u_k)$. Then for each solution of (5), (6) one has $S(\bar{y}(t), \bar{u}(t)) \geq 0$. Moreover, for any $\mu, \nu > 0$ there exists $\rho > 0$ such that $S(\bar{y}(t), \bar{u}(t)) > \rho$ whenever $\nu > \max_{1 \leq j,k \leq N} |y_k(t) - y_j(t)| > \mu$ and $t \geq 0$.

Proof. Let $\xi_{jk}(t) = a_{jk}(G(t))\varphi_{jk}(t, y_k(t) - y_j(t))$ and $\eta_{jk}(t) = a_{jk}(G(t))(y_k(t) - y_j(t))$. Accordingly to (6), one has $u_j(t) = \sum_{k=1}^{N} \xi_{jk}(t)$. Since $\xi_{jj} = 0$, one has $|u_j(t)|^2 \leq (N - 1) \sum_{k=1}^{N} |\xi_{jk}(t)|^2$ by the Cauchy-Schwartz inequality. Also it is evident that $\sum_{j,k=1}^{N} \xi_{jk} y_j = \sum_{j=1}^{N} u_j y_j$. Indeed, if Assumption 3a holds one has $\xi_{jk} = -\xi_{kj}$ and thus $\sum_{j,k=1}^{N} \xi_{jk} y_j = -\sum_{j,k=1}^{N} \xi_{kj} y_k = -\sum_{k=1}^{N} u_k y_k$. If Assumption 3b holds, take without loss of generality $w_{jk}(t) = 0$ if $k$ is not connected to $j$ in $G(t)$. Then $\sum_{j,k=1}^{N} \xi_{jk} y_k = \sum_{j,k=1}^{N} w_{jk} y_k (y_k - y_j) = \sum_{j,k=1}^{N} w_{jk} y_k^2 - \sum_{j,k=1}^{N} w_{jk} y_k y_j = \sum_{j,k=1}^{N} w_{jk} y_k^2 - \sum_{j,k=1}^{N} w_{kj} y_k y_j = \sum_{j,k=1}^{N} \xi_{jk} y_k = \sum_{k=1}^{N} u_k y_k$.

Due to (8) one has that either $(\varphi_{jk} - \alpha \sigma)(\beta \sigma - \varphi_{jk}) \geq 0$ (if $\beta < \infty$) or $(\varphi_{jk} - \alpha \sigma)\sigma \leq 0$ (if $\beta = \infty$). In both cases the inequality $\sigma(\varphi_{jk}(t, \sigma) - \delta|\varphi_{jk}(t, \sigma)|^2 - \gamma|\varphi_{jk}(t, \sigma)|^2)^2 \geq 0$ is immediate. If $y$ is bounded and separated from 0, the same is true for the left part of the latter inequality by definition of $S[\alpha; \beta]$. Taking $\sigma = y_k - y_j$, one obtains

$$\xi_{jk}(y_k - y_j) - \alpha \xi_{jk}(t)|y_k - y_j|^2 - \gamma \xi_{jk}^2 \geq 0. \quad (11)$$

Moreover, for any $\mu, \nu > 0$ there exists $\sigma(\mu, \nu) > 0$ such that if $k$ is connected to $j$ in $G(t)$ and $\nu > |y_k(t) - y_j(t)| > \mu$ then the left side of (11) is greater than $\sigma$. Take the sum of the inequalities (11) over all $j, k$ and notice that $\sum_{j,k=1}^{N} a_{jk}(t)|y_k - y_j|^2 \geq N^{-1} \sum_{j,k=1}^{N} |y_k - y_j|^2$ by the definition of $\lambda_2(G(t))$ (2). Since $\sum_{j=1}^{N} u_j = 0$, $\sum_{j,k=1}^{N} \xi_{jk}(y_k - y_j) = -2 \sum_{j=1}^{N} u_j y_j$ and $\sum_{j,k=1}^{N} \xi_{jk}^2 \leq (N - 1)^{-1} \sum_{j=1}^{N} u_j^2$, one obtains that $-2(2 \sum_{j=1}^{N} u_j y_j + N^{-1} \delta \sum_{j,k=1}^{N} (y_j - y_k)^2 + \frac{1}{\lambda - 1} \sum_{j=1}^{N} u_j^2) = N^{-1} S(\gamma(t), \bar{u}(t)) \geq 0$. Whenever $\nu > \max_{1 \leq j,k \leq N} |y_k(t) - y_j(t)| > \mu$, one has that $|y_k(t) - y_j(t)| > \mu/(2N)$.
for some \(j, k\) such that \((k,j)\in E(t)\) since \(G(t)\) is strongly connected. Thus for 
\[\rho(\mu, \nu) = N \varepsilon(\mu/(2N), \nu)\]
one obtains \(S(\bar{y}(t), \bar{u}(t)) > \rho(\mu, \nu)\) q.e.d.

**Lemma 8.** Under the assumptions (a) and (b) of Theorem 4 there exists a symmetric matrix \(H = H^* > 0\) such that the function \(V(\bar{x}) = \sum_j H(x_j)\) satisfies the inequality 
\[\dot{V}(\bar{x}(t)) + S(\bar{y}(t), \bar{u}(t)) \leq 0\]
for some \(\rho\) satisfies the inequality. \(\dot{V}(\bar{x}(t)) + S(\bar{y}(t), \bar{u}(t)) \leq 0\) for any solution of \((5)\).

Proof. Calculating \(\dot{V}(\bar{x}) = 2 \sum_j (x_j - x_j^*)^* H(x_j - x_j)\) one shows that the inequality \(\dot{V} + S \leq 0\) is equivalent to LMI as follows: 
\[2x^*H(Ax + Bu) + F(Cx, u) \leq 0\]
for any \(x \in \mathbb{R}^d, u \in \mathbb{R}\) where \(F\) is defined in Lemma 7. By the KYP lemma ("frequency theorem") \([5]\) the solvability of that LMI is equivalent to the frequency-domain inequality as follows: \(F(C(i\omega I - A)^{-1}Bu, u) \leq 0\) for any \(u \in \mathbb{C}\) and \(\omega \in \mathbb{R}\) such that \(\det(i\omega I - A)^{-1} \neq 0\). The latter inequality is nothing else than \((10)\). We show now that, if exists, the matrix \(H\) is positive definite.

Let \((\bar{x}_j^+, u_j^+, y_j^+)\) \((j = 1, \ldots, N)\) be a solution of the linear stationary system consisting of the agents \((5)\) and the protocol with all-to-all communication and \(\varphi_{jk}(t, y) = \mu y\) where \(\mu\) is from the assumption (a). This protocol satisfies all of the assumptions of Lemma 7 thus \(S(\bar{y}^+, \bar{u}^+) \geq 0\), achieving the equality only for \(y_j^+ = \ldots = y_N^+\). In the same time \(\lim_{\xi \to +\infty} |x_j^+(t) - x_k^+(t)| = 0\). Integrating the inequality \(\dot{V}(\bar{x}^+(t)) + S(\bar{y}^+, \bar{u}^+) \leq 0\) one obtains that \(\dot{V}(\bar{x}^+(0)) \geq 0\). If \(V(\bar{x}^+(t)) = 0\) then \(y_j^+(t) = \ldots = y_N^+(t)\) for any \(t \geq 0\), so \(u_j(t) = 0\) and by the observability \(x_j^+(0) = \ldots = x_N^+(0)\), which implies \(H > 0\).

**Proof of Theorem 4.** Under assumptions of the Theorem 4, consider the function \(V(\bar{x})\) from Lemma 8. For any solution of the system \((5),(6)\) one obtains

\[0 \leq V(\bar{x}(t)) \leq V(\bar{x}(0)) - \int_0^t S(\bar{y}(\xi), \bar{u}(\xi))d\xi \leq V(\bar{x}(0)),\]

(12)
The inequality (12) proves that \(V(\bar{x}(t))\) is bounded. To prove the output consensus, assume on the contrary that \(\zeta > 0\) and a sequence \(t_n \uparrow +\infty\) exist such that \(\max_{j,k} |y_j(t_n) - y_k(t_n)| > 2\zeta\). Since \(\bar{x}_j(t)\) are bounded, \(\delta > 0\) exists such that \(\max_{j,k} |y_j(t_n) - y_k(t_n)| > \zeta\) whenever \(t \in \Delta_n := (t_n - \delta, t_n + \delta)\). By Lemma 7 \(\inf_{\xi \in \Delta_n} S(\bar{y}(\xi), \bar{u}(\xi)) > 0\) which contradicts to (12).

### 3.3 Examples.

Below we apply the result of Theorem 4 to the agents with special dynamics.

**Passive agents.** Assume the positive realness condition

\[\text{Re} W_y(i\omega) \geq 0, \quad \forall \omega \in \mathbb{R},\]

(13)
to be valid. Combined with the assumption (a) from Theorem 4, (13) means the passivity property \([2]\) of the agents. As the condition (13) implies (10) for any \(\theta\), the following result is immediate from Theorem 4.
Theorem 9. Let the Assumption 3 be valid, \( \varphi_{jk} \in S[0; +\infty] \), \( A - \mu BC \) is Hurwitz for some \( \mu > 0 \) and also (13) holds. Then the output consensus (and the state consensus under the assumption (7)) is provided.

Theorem 9 extends a number of known results on low-order consensus models. For the first-order integrator agent \( \dot{y}_j = u_j \in \mathbb{R} \) (with \( W_y(\lambda) = \lambda^{-1} \)) it was proved in [2, 7, 10] etc. A more complicated example is the network of harmonic oscillators governed by the equations \( \dot{q}_j = y_j \in \mathbb{R}, \dot{y}_j = -\omega_j^2 q_j + u_j \) (the output \( y_j \) stands for the angular velocity and \( W_y(\lambda) = \lambda/(\lambda^2 - \omega_j^2) \)). It evident that the feedback \( u_j = -\mu_j y_j \) stabilizes the agent for any \( \mu > 0 \) and \( \text{Re} W_y(i\omega) = 0 \), thus (13) holds. Theorem 9 for this type of agents extends the result by W.Ren [9], concerning the case of linear balanced protocol (Assumption 4b) with constant gains.

**Strictly passifiable agents.** A natural generalization of passive agents is passifiable or agents which may be rendered passive by applying a feedback. We consider the case of strictly passifiable [4] agents (5) which means that \( CB > 0 \) and the polynomial \( \psi(\lambda) = \det(\lambda I - A)^{-1}W_y(\lambda) \) is Hurwitz (minimum-phase condition). In particular [4], for any sufficiently large \( k > 0 \) the matrix \( A - kBC \) is Hurwitz. It appears that any protocol satisfying Assumption 3 with sufficiently "strong" couplings provides the output consensus.

Theorem 10. Suppose that the agents are strictly passifiable, Assumption 3 holds, \( \varphi_{jk} \in S[\alpha; +\infty] \) and \( \alpha \geq K \theta^{-1} \), where \( \theta = \min_2 \lambda_2(G(t)) \) and \( K > 0 \) satisfies

\[
\text{Re} W_y(i\omega) + K |W(i\omega)|^2 \geq 0
\]

(such \( K \) exists). Then the protocol (6) provides the output consensus.

Notice first that (14) holds for \( K \) sufficiently large. Indeed, the function \( f(\omega) = \text{Re} W_y(i\omega)/|W_y(i\omega)|^2 = \text{Re} \left\{ \psi(-i\omega)^{-1} \det(-i\omega I - A) \right\} \) is continuous at any \( \omega \in \mathbb{R} \) and \( f(\omega) \to -CAB/(CB)^2 \) as \( \omega \to \pm \infty \), so \( f(\omega) \) is bounded from below. Proof of Theorem 10 is immediate from Theorem 4 for \( \beta = \infty \) (thus \( \delta = \alpha, \gamma = 0 \)): as noticed above, the assumption (a) of Theorem 4 is valid for \( \mu \) sufficiently large and the frequency-domain condition (3) holds because \( \delta \theta \geq K \).

**Second-order integrators.** The latter example may be further detailed for the case of double integrators

\[
\ddot{z}_j = u_j, \quad y_j = q_0 z_j + q_1 \dot{z}_j, \quad j = 1, 2, \ldots, N, \quad q_0, q_1 > 0.
\]

Consensus problem for double integrators (see [1, 10] etc.) has numerous applications in motion control. Since \( W'_y(\omega) = q_0 (\omega)^{-2} + q_1 (\omega)^{-1} \), the condition (14) is rewritten as \( K q_0^2 \omega^{-4} + (K q_1^2 - q_0) \omega^{-2} \geq 0 \), which gives a corollary as follows.

Corollary 11. Suppose the agents (15) apply the protocol (6) such that Assumption 3 holds and \( \varphi_{jk} \in S[\alpha; +\infty], \alpha > 0 \). Let \( K = \alpha \min_2 \lambda_2(G(t)) \) satisfies the inequality \( K q_1^2 \geq q_0 \). Then the output consensus is achieved.
Bibliography


