ROBUST SYNCHRONIZATION OF LINEAR MULTI-AGENT SYSTEMS WITH ADDITIVE UNCERTAINTY

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EXTENDED ABSTRACT. In this paper we will deal with the problem of robust synchronization of uncertain multi-agent networks. Given a network with for each of the agents identical nominal linear dynamics, we allow uncertainty in the form of additive perturbations of the transfer matrices of the nominal dynamics. The perturbations are assumed to be stable and bounded in $\mathcal{H}_\infty$-norm by some a priori given desired radius of synchronization. We will derive state space formulas for observer based dynamic protocols that achieve synchronization for all perturbations bounded by this desired radius. It is shown that a protocol achieves robust synchronization if and only if each controller from a related finite set of feedback controllers robustly stabilizes a given, single linear system. Our protocols are expressed in terms of the maximal real symmetric solutions of certain algebraic Riccati equations, and also involve weighting factors that depend on the one but smallest and largest eigenvalue of the Laplacian. In the full paper corresponding to this abstract it is shown that within the class of such dynamic protocols, a guaranteed achievable synchronization radius can be obtained that is proportional to the quotient of the one but smallest and the largest eigenvalue of the Laplacian.

1. Robust synchronization. We consider multi-agent networks with $p$ agents, where the underlying network graph is assumed to be an undirected, connected graph whose Laplacian is denoted by $L$. The dynamics of agent $i$ is given by the nominal system

$$\dot{x}_i = Ax_i + Bu_i, \quad y_i = Cx_i. \quad (1.1)$$

Thus, the nominal dynamics of each agent is represented by one and the same linear input-output system. Throughout this paper, we assume that the pair $(A,B)$ is stabilizable, and the pair $(C,A)$ is detectable. Each state $x_i$ takes its values in $\mathbb{R}^n$, the input $u_i$ and output $y_i$ take their values in $\mathbb{R}^m$ and $\mathbb{R}^q$ respectively. We now take the point of view that the dynamics of any of the agents is uncertain, and can be given by any system in a ball around the nominal system. In this paper we will quantify this by additive perturbations of the agent transfer matrices. In particular, as $G(s) = C(sI - A)^{-1}B$ represents the nominal system for agent $i$, we will consider perturbations $G(s) + \Delta_i(s)$, where $\Delta_i \in \mathcal{R}\mathcal{H}_\infty$. If we temporarily write

$$\Delta_i(s) = C_{\Delta_i}(sI - A_{\Delta_i})^{-1}B_{\Delta_i} + D_{\Delta_i},$$

this means that the dynamics of agent $i$ is perturbed to the system obtained by interconnecting

$$\dot{x}_i = Ax_i + Bu_i, \quad y_i = Cx_i + d_i, \quad z_i = u_i \quad (1.2)$$

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with
\[
\dot{\xi}_i = A\Delta_i \xi_i + B\Delta_i z_i, \quad d_i = C\Delta_i \xi_i + D\Delta_i z_i.
\] (1.3)

We allow all such perturbations with transfer matrix \(\Delta_i \in \mathbb{R}\mathcal{H}_\infty\) with \(\|\Delta_i\|_\infty \leq \eta\), where \(\eta > 0\) is a given uncertainty radius. Thus, the system describing the dynamics of agent \(i\) is any system with transfer matrix of the form \(G + \Delta_i\) with \(\|\Delta_i\|_\infty \leq \eta\).

Instead of explicitly writing out equations of the form (1.3) for the perturbation, in the sequel we will simply write
\[
d_i = \Delta_i z_i.
\]

**Definition 1.1.** Given a desired synchronization radius \(\eta > 0\), the problem of robust synchronization is to find a dynamic protocol such that for all \(i\) and for all \(\Delta_i \in \mathbb{R}\mathcal{H}_\infty\) with \(\|\Delta_i\|_\infty \leq \eta\), the network is synchronized, i.e. for all \(i, j = 1, 2, \ldots, p\) we have \(x_i(t) - x_j(t) \to 0\) and \(w_i(t) - w_j(t) \to 0\) as \(t \to \infty\).

For the purpose of robust synchronization we consider protocols of the form
\[
\dot{w}_i = Aw_i + BF \sum_{j \in N_i} \frac{1}{N}(w_i - w_j) + G(\sum_{j \in N_i} \frac{1}{N}(y_i - y_j) - Cw_i), \quad u_i = Fw_i.
\] (1.4)

Here \(N\) is a positive real number that, next to \(F\) and \(G\), needs to be determined. The following theorem states that in order to obtain a protocol that robustly synchronizes the network with synchronization radius \(\eta > 0\), we should find a positive real number \(N\), and gain matrices \(F\) and \(G\) such that each element of a collection of \(p - 1\) controllers robustly stabilizes a (single) system with stability radius \(\eta\):

**Theorem 1.2.** Let \(\eta > 0\). The following two statements are equivalent:

1. The dynamic protocol (1.4) synchronizes the network with perturbed agent dynamics
   \[
   \dot{x}_i = Ax_i + Bu_i, \quad y_i = Cx_i + d_i, \quad z_i = u_i, \quad d_i = \Delta_i z_i
   \]
   for all \(\Delta_i \in \mathbb{R}\mathcal{H}_\infty\) with \(\|\Delta_i\|_\infty \leq \eta\),

2. the perturbed linear system
   \[
   \dot{x} = Ax + Bu, \quad y = Cx + d, \quad z = u, \quad d = \Delta z
   \] (1.5)
   is stabilized for all \(\Delta \in \mathbb{R}\mathcal{H}_\infty\) with \(\|\Delta\|_\infty \leq \eta\) by all \(p - 1\) controllers
   \[
   \dot{w} = Aw + Bu + G(y - Cw), \quad u = \frac{1}{N}\lambda_i Fw, \quad i = 2, 3, \ldots, p.
   \] (1.6)

**2. Robustly synchronizing protocols.** Due to space limitations, we only consider the case that the matrix \(A\) does not have eigenvalues on the imaginary axis, and omit the general case here. Associated with \((A, B, C)\) we consider the following algebraic Riccati equation
\[
A^TP + PA - \gamma PBB^TP = 0,
\] (2.1)

together with the strict Riccati inequality
\[
AQ + QA^T - QC^TCQ < 0.
\] (2.2)
In (2.1), $\gamma$ is a positive real number that will be specified later. Let $P(\gamma)$ be the maximal real symmetric solution of (2.1). Then $P(\gamma) \geq 0$. Also, $A - \gamma BB^T P(\gamma)$ is Hurwitz (this uses the assumption that $A$ has no eigenvalues on the imaginary axis). Let $Q > 0$ be any positive definite solution to (2.2). It is easily seen that such $Q$ exists.

In the following, $\lambda_2$ and $\lambda_p$ are the one but smallest and largest eigenvalue of the Laplacian $L$. We assume that $\lambda_2 > 0$. The following theorem yields a robustly synchronizing dynamic protocol for the perturbed multi-agent network. The synchronization radius that we obtain depends on the spectral radius $\rho(P(\gamma)Q)$ of the product of $P(\gamma)$ and $Q$ as given by (2.1) and (2.2):

**Theorem 2.1.** Consider the network with $p$ agents, with perturbed agent $i$ given by

$$\dot{x}_i = Ax_i + Bu_i, \quad y_i = Cx_i + d_i, \quad z_i = u_i, \quad d_i = \Delta_i z_i$$

Choose $N$ any positive real number such that

$$N > \frac{\lambda_2}{\lambda_p}, \quad (2.3)$$

equivalently $(\frac{\lambda_p}{N^2})^2 < \frac{\lambda_2}{N}$. Next, choose $\gamma$ such that

$$(\frac{\lambda_p}{N^2})^2 < \gamma < \frac{\lambda_2}{N}. \quad (2.4)$$

Then, let $P(\gamma)$ be the maximal real symmetric solution of (2.1) and let $Q > 0$ be any solution of (2.2). Let $\eta$ be any positive real number such that

$$\eta < \frac{1}{\sqrt{\rho(P(\gamma)Q)}}. \quad (2.5)$$

Define

$$F := -B^T P(\gamma) \quad (2.6)$$

$$G := (I - \eta^2 P(\gamma))^{-1} QC^T \quad (2.7)$$

Then the dynamic protocol

$$\dot{w}_i = Aw_i + BF \sum_{j \in N_i} \frac{1}{N}(w_i - w_j) + G(\sum_{j \in N_i} \frac{1}{N}(y_i - y_j) - Cw_i),$$

$$u_i = Fw_i,$$

synchronizes the network for all perturbations $\Delta_i \in RH_{\infty} (i = 1, 2, \ldots, p)$ with

$$\|\Delta_i\|_{\infty} \leq \eta. \quad (2.8)$$
REFERENCES